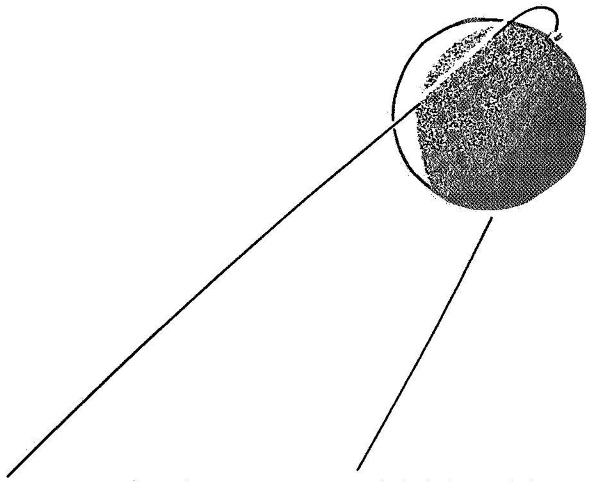

LIAPUNOV FUNCTIONS FOR FEEDBACK SYSTEMS CONTAINING A SINGLE TIME-VARYING NONLINEARITY

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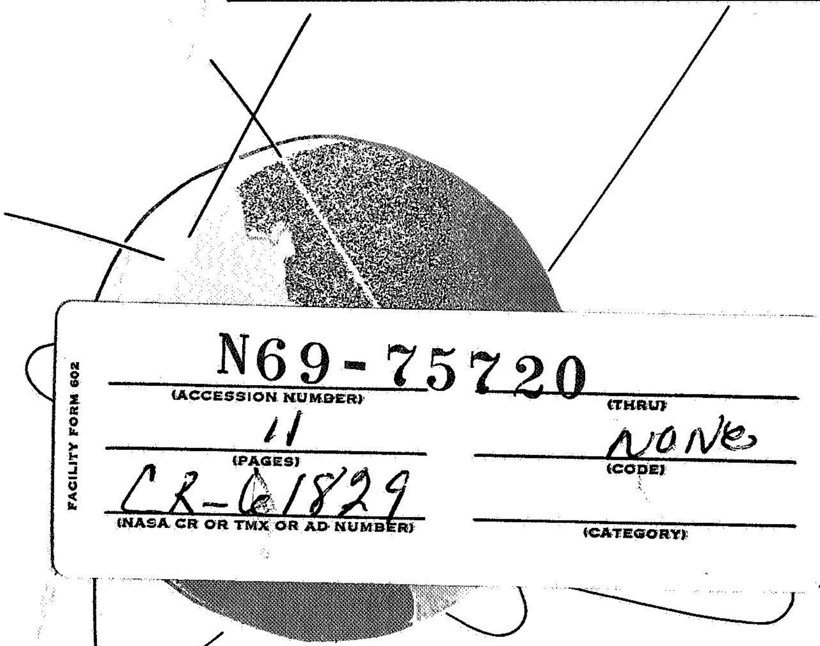

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It will also be assumed that the basic Popov condition,

$$\mathrm{Re}(1 + i\omega q)c'[A - i\omega I]^{-1}b + \frac{1}{K} \geq \delta > 0 \qquad (1.3)$$

is satisfied for all real ω by two real numbers q and $K(K > 0)$.

2. Theorem.

Consider system (1.1) where A is Hurwitz and the nonlinearity satisfies the conditions

$$0 \leq \Phi(\sigma,t) \leq \Psi(\sigma) \quad (\Psi(0) = 0) \qquad (2.1)$$

$$\Phi_\sigma^2 \leq K(\sigma\Phi_\sigma - q\Phi_t) \qquad (2.2)$$

for some $K > 0$, $q \geq 0$, and some continuous function Ψ for all σ and $t \geq 0$. Also consider a function V of the form

$$V(x,t) = x'Bx + q\Phi(\sigma,t) \qquad (2.3)$$

where B is a real nxn matrix.

A. If the Popov condition (1.3) is satisfied, there exists a real positive definite matrix B such that $\dot{V}(x,t)$ for system (1.1) is negative definite.

B. $V(x,t)$ is then a Liapunov function which assures global asympotic stability of the equilibrium by the Barbasin-Krasovskii Theorem [4] for all $\Phi(\sigma,t)$ satisfying (2.1) and (2.2).

C. If (A,b) is completely controllable, there is an effective construction procedure for $V(x,t)$.

D. The preceding statements remain valid for $q \leq 0$ if (2.1) is replaced by

$$\Psi(\sigma) \leq \Phi(\sigma,t) \leq \frac{K}{2}\sigma^2 \quad (\Psi(0) = 0) \tag{2.4}$$

Comment: For the time-invariant case or for $q = 0$ conditions (2.1) and (2.2), or (2.4) and (2.2), reduce to a definition of the usual Popov sector; i.e.

$$0 \leq \sigma\Phi_\sigma \leq K\sigma^2$$

3. <u>Proof of Part A.</u>

The time derivative of $V(x,t)$ according to (1.1) is of the form

$$-\dot{V}(x,t) = x'Dx + \left(u'x - \sqrt{\gamma}\,\Phi_\sigma\right)^2 + \left(\sigma\Phi_\sigma - q\Phi_t - \Phi_\sigma^2/K\right) \tag{3.1}$$

where $\gamma \triangleq 1/K - qc'b$ and D and u are related to B by means of

$$A'B + BA = -uu' - D \tag{3.2}$$

$$Bb + \frac{1}{2} qA'c + \frac{1}{2} c = \sqrt{\gamma}\, u \tag{3.3}$$

Lemma (K. Meyer, [5]): Let A be a real nxn matrix all of whose characteristic roots have negative real parts, γ be a real non-negative number and b and h be any two real n-vectors. If

$$\gamma + 2 \operatorname{Re} h'\left[i\omega I - A\right]^{-1} b > 0$$

for all real ω, then there exist two real positive definite matrices B and D and a real n-vector u such that

$$A'B + BA = -uu' - D$$

$$Bb - h = \sqrt{\gamma}\, u$$

Setting $2h = -qA'c - c$, the Lemma refers to relations (1.3), (3.2), and (3.3). Since the last two terms of (3.1) are non-negative, $-\dot{V}(x,t) \geq x'Dx$ and the Lemma implies there exists a positive definite matrix B such that $\dot{V}(x,t)$ is negative definite.

4. Proof of Part B.

Condition (2.1) implies

$$x'Bx \leq V(x,t) \leq x'Bx + q\Psi(\sigma)$$

and therefore $V(x,t)$ is positive definite, radially unbounded, and decresent. Since $\dot{V}(x,t)$ is negative definite for all $\Phi(\sigma,t)$ satisfying (2.1) and (2.2), the Barbasin-Krasovskii Theorem [4] assures global asymptotic stability of the equilibrium of (1.1).

5. Proof of Part C[1].

Define $D = \varepsilon D_1 (\varepsilon > 0)$ where D_1 is any given real symmetric positive definite nxn matrix, and define $m(z) = [zI - A]^{-1}b$. The Popov condition (1.3) may be written[2]

$$\gamma + m^*(i\omega)h + h'm(i\omega) \geq \delta > 0 \tag{5.1}$$

Let ν denote the greatest lower bound of $m^*(i\omega)h + h'm(i\omega)$ and μ denote the least upper bound[3] of $m^*(i\omega)D_1 m(i\omega)$ over

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[1] The proof of Part C closely follows the sufficiency proof of a lemma found in [6], p. 115.

[2] $m^*(i\omega)$ denotes the conjugate transpose of $m(i\omega)$.

[3] These bounds exist since $m(i\omega) \to 0$ as $\omega \to \infty$ and $m(i\omega)$ is a continuous function of ω.

all real ω . Then $\gamma + \nu > 0$ and

$$\gamma + m^*(i\omega)h + h'm(i\omega) - \varepsilon m^*(i\omega)D_1m(i\omega) \geq \gamma + \nu - \varepsilon\mu$$

If ε is chosen sufficiently small; i.e.,

$$0 < \varepsilon < \frac{\gamma+\nu}{\mu} \tag{5.2}$$

this implies

$$\gamma + m^*(i\omega)h + h'm(i\omega) - \varepsilon m^*(i\omega)D_1m(i\omega) > 0$$

As shown by Kalman [3], the left side of this relation may be written as $\theta(i\omega)\theta(-i\omega)/|i\omega I-A||-i\omega I-A|$ where θ is a real polynomial of degree n with leading coefficient $\sqrt{\gamma}$. Following the work of Kalman [3], a possible construction of the vector u may be defined by

$$u'm(z) = \frac{\sqrt{\gamma}\ |zI-A|-\theta(z)}{|zI-A|} \tag{5.3}$$

provided (a,b) is completely controllable.
Having defined D and u , B may be found by means of (3.2) and is necessarily positive definite since A is Hurwitz. Matrix B does satisfy (3.3) since, by (5.3),

$$\gamma + m^*(i\omega)h + h'm(i\omega) - \varepsilon m^*(i\omega)D_1 m(i\omega)$$

$$= \left(\sqrt{\gamma} - u'm(i\omega)\right)\left(\sqrt{\gamma} - m^*(i\omega)u\right)$$

and therefore, by (3.2),

$$\mathrm{Re}\; m^*(i\omega)(Bb - h - \sqrt{\gamma}\, u) = 0$$

for all real ω .

6. Proof of Part D.

If $q \leq 0$ and condition (2.4) is satisfied rather than (2.1), $\dot{V}(x,t)$ remains negative definite and a positive definite matrix B exists as already shown. However, the last term of $V(x,t)$, given by (2.3), is no longer non-negative and $V(x,t)$ must be reconsidered.

Since the Popov condition (1.3) is satisfied, the Nyquist criterion assures asymptotic stability of (1.1) for all linear time-invariant characteristics $\phi(\sigma,t) = h\sigma$, $0 \leq h \leq K + \varepsilon$ for sufficiently small $\varepsilon > 0$. Since $\dot{V}(x,t)$ is also negative definite for all such characteristics, Liapunov's First Instability Theorem [4] implies that $V(x,t)$ for all such characteristics is non-negative. In particular,

$$x'Bx - (-q)\,\frac{K+\varepsilon}{2}\,\sigma^2 \geq 0 \qquad \text{for all } x$$

and therefore

$$x'Bx + q \frac{K}{2} \sigma^2 > 0 \qquad \text{for all } x \neq 0$$

Writing $V(x,t)$ for a general characteristic as

$$V(x,t) = x'Bx + q \frac{K}{2} \sigma^2 + (-q)\left(\frac{K}{2} \sigma^2 - \Phi(\sigma,t)\right)$$

the last term is non-negative by (2.4) and

$$x'Bx + q \frac{K}{2} \sigma^2 \leq V(x,t) \leq x'Bx + q\Psi(\sigma)$$
$$\text{for all } x \text{ and } t \geq 0.$$

Since V is positive definite, radially unbounded, and decresent, (B) is correct for $q \leq 0$ if (2.1) is replaced by (2.4).

The proofs of (A) and (C) are not affected by these changes and need not be reconsidered.

7. Conclusions.

The existence of Liapunov functions of the form "quadratic plus integral of the nonlinearity" is found to be assured by the satisfaction of the original Popov inequality even when the nonlinearity is time-varying, provided the Popov sector is appropriately modified. The Kalman construction procedure is found to also remain valid. The modified sector reduces to the usual Popov sector in the time invariant case.

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